Limit Comparison Test

Let $\sum_{\text{k}} a_k$ be a series with $\underline{a_k \geq 0 \text{ for all } k}$.

Select a series $\sum_k b_k$. If $\lim_{n\to\infty} \frac{a_n}{b_n} = c > 0$,

then both series converge or both series diverge.

NOTE: *Use one of the series you KNOW converges or diverges (geometric, p-series, etc.).*
This test is a good alternative to the comparison test.

Example: Does the series converge?

(A) $\sum_{k=1}^{\infty} \frac{k+1}{k^3+4}$, $a_k = \frac{k+1}{k^3+4} \geq 0$ for all $k \geq 1$.

→ intuition: looks "almost" like a p-series with $p = 2 > 1$, so we should expect the series to converge.

→ to show convergence, apply the limit comparison test.

Take $b_k = \frac{k}{k^3} = \frac{1}{k^2}$.

Then we know that $\sum_{k=1}^{\infty} b_k$ converges

(p-series with p=2). (*)

$\frac{1}{\frac{1}{k^2}} = k^2$

→ compute the limit:

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{(k+1)}{(k^3+4)} \cdot \boxed{k^2} = 1 > 0$$

→ by the limit comparison test both $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ converge or diverge

→ so by (*) the original series converges

Example: Does the series converge?

(B) $\sum_{k=1}^{\infty}\frac{k}{\sqrt{k^3+1}} = S = \sum_{k=1}^{\infty} a_k$, where $a_k = \frac{k}{\sqrt{k^3+1}}$,

almost like $\frac{k}{k^{3/2}} = \frac{1}{\sqrt{k}}$

$a_k \geq 0$ for all $k \geq 1$, so apply the limit comparison test.

→ intuition: S looks "approximately" like a p-series with $p = 3/2 - 1 = \frac{1}{2}$. So we should expect that S diverges.

→ LCT: take $b_k = \frac{1}{\sqrt{k}}$, so that

$\sum_{k=1}^{\infty} b_k$ diverges (p-series with $p = 1/2 \leq 1$)

$\longrightarrow$ compute the limit:

$$\lim_{N\to\infty} \frac{a_N}{b_N} = \lim_{N\to\infty} \frac{\boxed{N^{1/2}} \cdot N}{\sqrt{N^3+1}}$$

$\frac{1}{b_N}$ $\qquad a_N$

$\longrightarrow \sqrt{N^3\left(1+\frac{1}{N^3}\right)} = N^{3/2}\sqrt{1+\frac{1}{N^3}}$

$$= \lim_{N\to\infty} \frac{\cancel{N^{3/2}}}{\cancel{N^{3/2}}\sqrt{1+\frac{1}{N^3}}} = 1 > 0$$

$\longrightarrow$ so by the LCT, both series diverge.

$\longrightarrow$ The series S diverges.

Challenge example: Does the series converge?

$$S = \sum_{n=2}^{\infty} \frac{e^{3n}}{e^{6n} + 16}$$

→ apply the integral test with

$$f(x) = \frac{e^{3x}}{e^{6x}+16},$$

- cts
- decreasing
- positive

→ integral test tells us the series conv. if $\int_2^\infty f(x)\,dx$ conv., and diverges otherwise

→ look at the indefinite integral: $\int f(x)\,dx$

$$\left(u\text{-sub: } u = e^{3x},\ du = 3e^{3x}\,dx,\ e^{6x} + 16 = u^2 + 16\right)$$

$$\int f(x)\,dx = \frac{1}{3}\int \frac{du}{u^2+16} = \frac{1}{48}\int \frac{du}{\left(\frac{u}{4}\right)^2+1}$$

$$\left(v\text{-sub: } v = \frac{u}{4},\ dv = \frac{du}{4}\right)$$

$$= \frac{4}{48}\cdot\int \frac{dv}{v^2+1} = \frac{1}{12}\tan^{-1}(v) + C$$

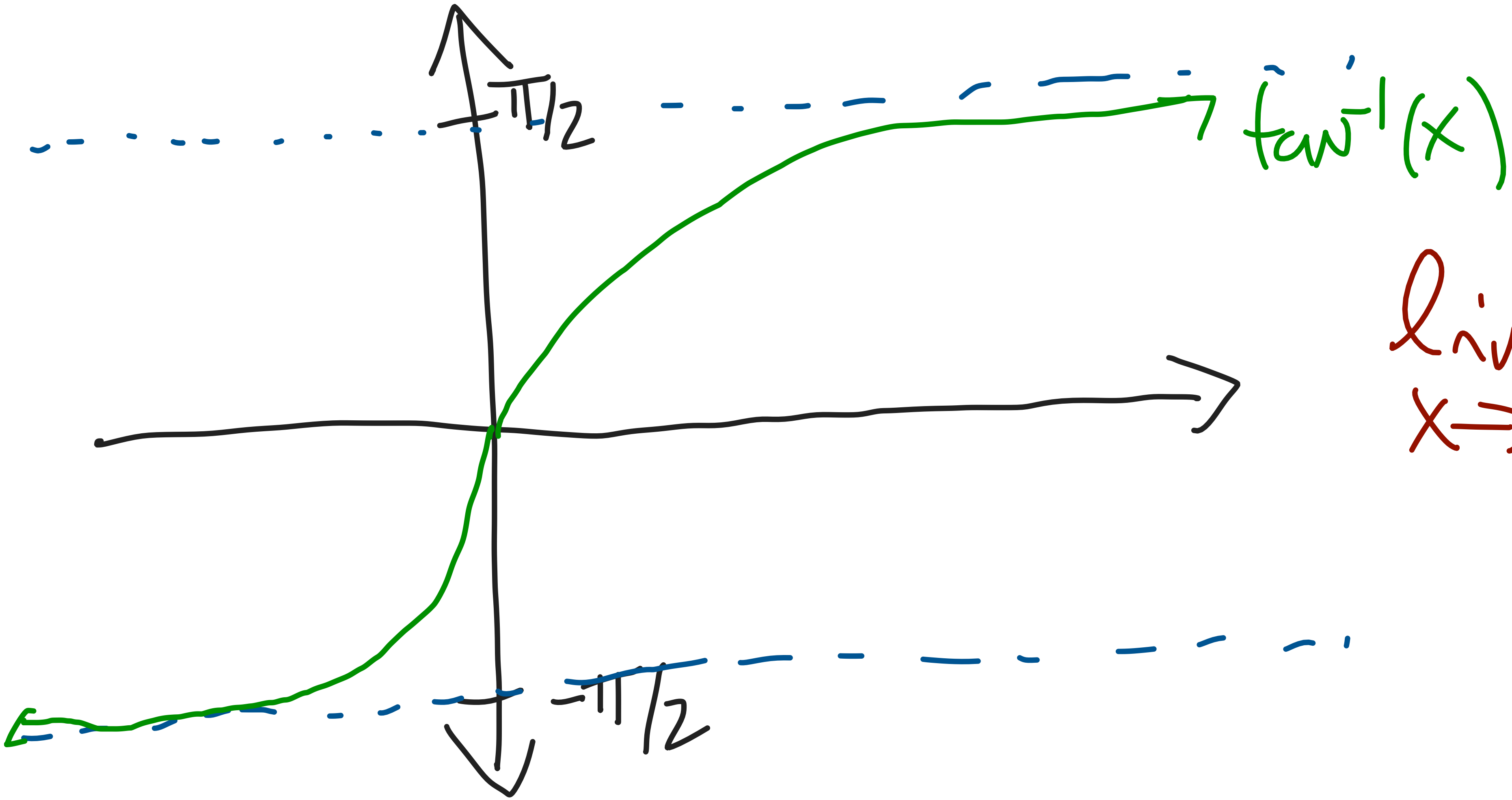
$$= \frac{1}{12}\tan^{-1}\left(\frac{u}{4}\right) + C$$

$$= \frac{1}{12}\tan^{-1}\left(\frac{e^{3x}}{4}\right)+C$$

$$\rightarrow \int_2^{\infty} f(x)\,dx = \lim_{b\to\infty} \frac{1}{12}\tan^{-1}\left(\frac{e^{3x}}{4}\right)\Bigg|_2^b$$

$$= \frac{1}{12}\left[\underbrace{\lim_{b\to\infty}\tan^{-1}\left(\frac{e^{3b}}{4}\right)}_{=\lim_{x\to\infty}\tan^{-1}(x)} - \underbrace{\tan^{-1}\left(\frac{e^{6}}{4}\right)}_{\text{some finite value}}\right]$$

(because $\frac{e^{3b}}{4}\to\infty$ as $b\to\infty$)



$$\lim_{x\to\infty} \tan^{-1}(x) = \pi/2$$

$\rightarrow$ So $\int_2^\infty f(x)\,dx$ is finite.

Therefore the series S converges by the integral test.

$$\left| \sum_{k=2}^{\infty} f(k) - \int_{2}^{\infty} f(x)\,dx \right| = C, \text{ some finite constant } C > 0.$$

Math 1552

Section 10.5: The Ratio and Root Tests for Infinite Series

Math 1552 lecture slides adapted from the course materials
By Klara Grodzinsky (GA Tech, *School of Mathematics*, Summer 2021)

Recap of last class:

- *Divergence test*: if the limit is not 0, the series diverges (nth term test)
- *Comparison test*: find a bigger series that converges or a smaller series that diverges (basic – BCT)
- *Integral test*: use with a function that has an "easy" antiderivative

Recap of last class:

- *Limit Comparison test*: pick a series that you know converges or diverges. (LCT)

(If the limit of the ratio of terms in your series to the given series approaches a finite, positive number, then both series either converge or diverge.)

Ratio Test

Let $\sum_{k=1}^{\infty} a_k$ be a series with all positive terms. (e.g., $a_k > 0$ for all $k \geq 1$)

Let $L = \lim_{n\to\infty} \frac{a_{n+1}}{a_n}$.

(a) If $L < 1$, then $\sum_{k=1}^{\infty} a_k$ coverges.

$0 \leq L < 1$

(b) If $L > 1$, then $\sum_{k=1}^{\infty} a_k$ diverges.

(c) If $L = 1$, then the test is *INCONCLUSIVE*!!!!

$\hookrightarrow$ apply another test to show convergence or divergence

Example 1:

Determine whether the next series converges or diverges. $S = \sum_{k=1}^{\infty} \frac{3^k}{k^2}$

$S = \sum_{k=1}^{\infty} a_k$, where $a_k = \frac{3^k}{k^2}$.

→ apply the ratio test (Note that $a_k > 0$ for all $k \geq 1$)

→ compute the limit

$$L = \lim_{N \to \infty} \frac{a_{N+1}}{a_N} = \lim_{N \to \infty} \frac{3^{N+1}}{(N+1)^2} \cdot \frac{N^2}{3^N}$$

$$= \lim_{n\to\infty} \frac{3n^2}{(n+1)^2} = \frac{3}{1} = 3 > 1.$$

$\rightarrow$ So by the ratio test, the infinite series S <u>diverges</u>.

Example 2:

Determine whether the next series converges or diverges. $S = \sum_{k=1}^{\infty} \frac{k \cdot 3^k}{(2k)!}$

$$S = \sum_{k=1}^{\infty} a_k, \text{ where } a_k = \frac{k \cdot 3^k}{(2k)!}$$

→ $a_k > 0$ for all $k \geq 1$, so we can apply the ratio test to check for convergence

→ compute the limit:

$$L = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)\cdot 3^{n+1}}{(2n+2)!} \cdot \frac{(2n)!}{n \cdot 3^n}$$

$3^{n+1} = 3^n \cdot 3$

- review of factorial function:

$$n! = n \cdot (n-1)! \text{ for all } n \geq 1$$

$0! = 1$

$1! = 1 = 1 \cdot 0!$

$2! = 2 \cdot 1! = 2$

$3! = 6$

$4! = 24$

- $(2n+2)! = (2n+2) \cdot (2n+1)!$

$$= (2n+2)(2n+1) \cdot (2n)!$$

So $L = \lim\limits_{n \to \infty} \dfrac{3(n+1)}{n(2n+2)(2n+1)} = 0$

→ by the ratio test with $L = 0 < 1$, the series S **converges**.

Root Test

Let $\sum_{k=1}^{\infty} a_k$ be a series with all positive terms. (e.g., we need that $a_k > 0$ for all $k \geq 1$)

Let $R = \lim_{n\to\infty} \sqrt[n]{a_n}$. $= \lim_{n\to\infty} a_n^{1/n}$

(a) If $R < 1$, then $\sum_{k=1}^{\infty} a_k$ coverges.

(b) If $R > 1$, then $\sum_{k=1}^{\infty} a_k$ diverges.

(c) If $R = 1$, then the test is *INCONCLUSIVE*!!!!

$\hookrightarrow$ Need to apply another test

Example:

Determine if the series converges or diverges. $S=\sum_{k=1}^{\infty}\left(1+\frac{2}{k}\right)^{k^2}$

$S=\sum_{k=1}^{\infty} a_k$, where $a_k=\left(1+\frac{2}{k}\right)^{k^2}$.

→ we have that $a_k > 0$ for all $k \geq 1$. So we can apply the root test

$\left(1+\frac{2}{N}\right)^{N^2 \cdot \frac{1}{N}}$

→ we compute the limit:

$$R=\lim_{N\to\infty} a_N^{1/N}=\lim_{N\to\infty}\left(1+\frac{2}{N}\right)^{N}=e^2>1$$

→ So by the root test with $R > 1$, we conclude that the infinite series S <u>diverges</u>.

If we had applied the ratio test instead:

$$L = \lim_{n\to\infty} \frac{a_{n+1}}{a_n}$$

$$= \lim_{n\to\infty} \frac{\left(1+\frac{2}{n+1}\right)^{(n+1)^2}}{\left(1+\frac{2}{n}\right)^{n^2}}$$

So the root test is substantially easier to apply!

Tips: which test to use when?

a.k.a. the n^{th} term test

- ALWAYS start with the divergence test.

if $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{n=1}^{\infty} a_n$ diverges.

otherwise, we need to apply another test.

Tips: which test to use when?

- ALWAYS start with the divergence test.
- Use the integral test if the function looks “easy” to integrate or can be solved with a u-substitution.

Tips: which test to use when?

- ALWAYS start with the divergence test.
- Use the integral test if the function looks "easy" to integrate or can be solved with a u-substitution.
- Use the harmonic series, geometric series, or p-series in the comparison and limit comparison tests.

Tips (continued)

- If you are unsure of which way the inequality may go, use the limit comparison test instead of the comparison test.

Tips (continued)

- If you are unsure of which way the inequality may go, use the limit comparison test instead of the comparison test.
- Use the root test when everything is raised to the k^{th} power.

Tips (continued)

- If you are unsure of which way the inequality may go, use the limit comparison test instead of the comparison test.
- Use the root test when everything is raised to the k^{th} power.
- Use the ratio test when you have factorials, or when no other test works.

Math 1552

Section 10.6: Alternating Series

Math 1552 lecture slides adapted from the course materials
By Klara Grodzinsky (GA Tech, *School of Mathematics*, Summer 2021)

Alternating Series Test

Let $\sum_k a_k$ be an alternating series.

$b_k \geq 0$ for all k.
Then an alternating series looks like
$\sum_k (-1)^k b_k$
$(a_k = (-1)^k b_k)$

(a) If $\sum_k |a_k|$ converges, then the series *converges absolutely*.

Alternating Series Test (cont.)

Let $\sum_{k} a_k$ be an alternating series.

, e.g., we don't have absolute conv. of the series

(b) If (a) fails, then if :

i) $\{|a_n|\}$ is a decreasing sequence, and

ii) $\lim_{n\to\infty} |a_n| = 0$,

then the series *converges conditionally*.

(c) Otherwise, the series *diverges*.

Example A:

Determine if the alternating series converges absolutely, converges conditionally, or diverges. $S = \sum_{k=1}^{\infty} (-1)^k \frac{1}{\sqrt{k+4}}$

$S = \sum_{k=1}^{\infty} (-1)^k b_k$, where $b_k = \frac{1}{\sqrt{k+4}}$.

$\rightarrow$ first, need to check for absolute convergence: does $\sum_{k=1}^{\infty} b_k$ converge?

apply the LCT to show that we do not get absolute convergence.